Lecture 3: Symmetries (CFT)

Content: Symmetries in Classical Field Theory, Noether's Theorem, Energy-Momentum Tensor, Charge Generators, Quantization Dogma.

Last day: Classical Field Theory, Classical Field quantization.

Symmetries in Classical Field Theory:

Suppose $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$; $\mu = 0,1,2,3$, $a = 1,2,\dots,N$

is a Lagrange density for some fields $\phi_a(x)$.

Consider an infinitessimal transformation

$$\phi_a'(x) = \phi_a(x) + \underbrace{X_a(\phi_a)}_{\text{infinitessimal}}. \tag{3.1}$$

This is called an (infinitessimal) Symmetry if it leaves equations of motion invariant:

$$\mathcal{L} \longmapsto \mathcal{L}(\phi_a', \partial_\mu \phi_a') = \mathcal{L}(\phi_a, \partial_\mu \phi_a) + \partial_\mu F^\mu \tag{3.2}$$

for some total derivative ∂_μ of some function F^μ.

Recall: $S[\phi_a] = \int d^4x\, \mathcal{L}' = \int d^4x \left(\mathcal{L} + \partial_\mu F^\mu\right) \qquad (3.3)$

Noether's Theorem:

Every continuous symmetry of a Lagrangian / Lagrange density implies the existence of a conserved current $j^\mu(x)$ where $\partial_\mu j^\mu = 0$.

Proof: Let $\overbrace{\delta\phi_a}^{\text{Infinitessimal}}$ be arbitrary change in the fields.

$$\Longrightarrow \delta\mathcal{L}(\phi_a, \partial_\mu\phi_a) = \mathcal{L}\left(\phi_a + \delta\phi_a,\ \partial_\mu(\phi_a + \delta\phi_a)\right) - \mathcal{L}(\phi_a, \partial_\mu\phi_a) \qquad (3.4)$$

FIRST ORDER

$$\overset{\mathcal{O}(\delta\phi_a)}{=} \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a) \qquad (3.5)$$

$$\overset{(\text{into } S[\phi_a])}{=} \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right]}_{\to 0}\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) \qquad (3.6)$$

However, when ϕ_a obeys the E-L equations, the first term in (3.6) vanishes. Thus we find

$$\delta\mathcal{L}(\phi_a, \partial_\mu \phi_a) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right) \qquad (3.7)$$

When $\delta\phi_a = X[\phi_a]$ (for infinitessimal symmetries)

THEN $\delta\mathcal{L} = \partial_\mu F^\mu$.

This implies

$$\partial_\mu F^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X[\phi_a] \right) \qquad (3.8)$$

which yields a conservation law

$$\partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X[\phi_a] - F^\mu \right\} = 0 \qquad (3.9)$$

or $\partial_\mu j^\mu = 0$, with

$$\boxed{j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X[\phi_a] - F^\mu} \qquad (3.10)$$

Conserved Current

$\partial_\mu j^\mu$ implies existence of conserved "charges"

Define: For any $V \subset \mathcal{M}_{1,3}$ measurable...

$$Q_V = \int_V d^3x \; j^0(x) \qquad (3.11)$$

Calculate: $$\frac{dQ_V}{dt} = \int d^3x \, \frac{\partial}{\partial t} j^0 \qquad (3.12)$$

continuity

$$= - \int_V d^3x \; \partial_k j^k \qquad (3.13)$$

divergence thm.

$$= - \int_{\partial V} \vec{j} \cdot d\vec{s} \qquad (3.14)$$

Let $V = \mathbb{R}^3$ (and assume $j \to 0$ as $|x| \to \infty$)

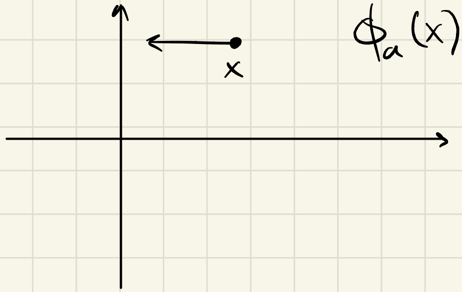
in which case $\frac{dQ_{\mathbb{R}^3}}{dt} = 0$.

We may also work backwards, in the sense that conserved charges generate symmetries.

Stress-Energy Tensor:

Example: Spacetime translations. 1 temporal / 3 spatial

Consider the active transformation $x^\mu \mapsto x^\mu - \varepsilon^\mu$.



$$\phi_a(x) \longrightarrow \phi_a'(x)$$

New field:

$$\phi_a'(x^\mu) = \phi_a(x^\mu + \varepsilon^\mu) \quad \text{Taylor Expand} \tag{3.15}$$

$$= \phi_a(x^\mu) + \underbrace{\varepsilon^\mu \partial_\mu \phi_a(x^\mu)}_{X[\phi_a]} \tag{3.16}$$

How does the Lagrange density change?

$$\mathcal{L}(\phi_a, \partial_\mu \phi_a) = \mathcal{L}(x) \tag{3.17}$$

$\hookrightarrow$ only a function of x^μ...

$$\Longrightarrow \mathcal{L}(x^{\mu'}) = \mathcal{L}(x) + \underbrace{\varepsilon^\mu \partial_\mu \mathcal{L}}_{\partial_\mu F^\mu} \tag{3.18}$$

4 symmetries $\Rightarrow$ 4 conserved currents $\Rightarrow$ 16 quantities

Consider

$$\varepsilon^\mu \in \{\underbrace{(1,0,0,0)}_{\hat{v}^0}, \underbrace{(0,1,0,0)}_{\hat{v}^1}, \underbrace{(0,0,1,0)}_{\hat{v}^2}, \underbrace{(0,0,0,1)}_{\hat{v}^3}\} \tag{3.19}$$

such that $\hat{v}^\mu_\nu = \delta^\mu_\nu$ (3.20)

Apply Noether's theorem to each of $\varepsilon \hat{v}^\mu$ in turn,

$$\boxed{j^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - F^\mu{}_\nu} \qquad (3.21)$$

$$\left(\underbrace{j^\mu{}_\nu \equiv T^\mu{}_\nu}_{\text{16 numbers}} \right)$$

STRESS-ENERGY TENSOR
(ENERGY-MOMENTUM)

Here, $F^\mu{}_\nu = \delta^\mu_\nu \mathcal{L}$ $\qquad (3.22)$

Proof: Recall that $\mathcal{L}(x') = \mathcal{L}(x) + \partial_\mu F^\mu$.

Thus $\delta\mathcal{L} = \partial_\mu F^\mu = \varepsilon^\mu \partial_\mu \mathcal{L}$.

Note that F^μ has a coordinate $\hat{v}^\mu$,

Then F^μ_ν has two coordinates $\hat{v}^\mu \hat{v}_\nu$.

Contracting along $\hat{v}^\mu$ implies

$$\partial_\mu F^\mu{}_\nu = \varepsilon \hat{v}^\mu \hat{v}_\nu \partial_\mu \mathcal{L}$$

$$= \varepsilon \delta^\mu{}_\nu \partial_\mu \mathcal{L}$$

($\hat{v}^\mu \hat{v}_\nu = \delta^\mu_\nu$ in tensor form)

But since ε, ∂_μ's arbitrary, we find

$$F^\mu_\nu = \delta^\mu_\nu \mathcal{L}. \qquad \blacksquare$$

These conserved currents correspond also to conserved charges:

$$\boxed{\partial_\mu T^\mu{}_\nu = 0.} \qquad (3.23)$$

$$\left(T^{\mu\nu} = \eta^{\nu\lambda} T^\mu{}_\lambda \right)$$

Corresponding conserved charges

Energy $\rightarrow$
$$E = \int d^3x\, T^{00} \qquad (3.24)$$

and
$$p^j = \int d^3x\, T^{0j} \qquad (3.25)$$

$\hookrightarrow$ momenta in j^{th} direction

Example (of an example):

Klein-Gordon Lagrange Density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\, \partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \qquad (3.26)$$

Easy to show that

$$T^{\mu\nu} = \partial^\nu\phi\,\partial^\mu\phi - \frac{1}{2}\delta^{\mu\nu}\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right) \qquad (3.27)$$

Proof: Note by definition

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} = \frac{1}{2}\frac{\partial}{\partial(\partial_\mu\phi_a)}\left\{\eta^{\lambda\kappa}\,\partial_\lambda\phi_a\,\partial_\kappa\phi_a\right\}$$

$$= \frac{\eta^{\lambda\kappa}}{2}\left\{\delta^\mu_\lambda\,\partial_\kappa\phi_a + \partial_\lambda\phi_a\,\delta^\mu_\kappa\right\}$$

$$= \frac{1}{2}\left(\partial^\mu\phi_a + \partial^\mu\phi_a\right)$$

$$= \partial^\mu\phi_a$$

Thus

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - \delta^{\mu\nu} \mathcal{L}$$

$$\Rightarrow \quad T^{\mu\nu} = \partial^\mu \phi_a \partial^\nu \phi_a - \frac{1}{2} \delta^{\mu\nu} \left(- \cdots \right)$$

$\blacksquare$

The energy is $\int d^3x \, T^{00}$:

$$E = \int d^3x \left([\partial^0 \phi]^2 - \frac{1}{2} [\partial^0 \phi]^2 + \frac{1}{2} m^2 \phi^2 \right) \tag{3.28}$$

$$= \int d^3x \, \mathcal{H}(x) \tag{3.29}$$

And

$$p^j = \int d^3x \, \dot{\phi} \, \partial^j \phi . \tag{3.30}$$

Lorentz and Poincare Transformations:

Infinitessimal:

$$x^\mu \longmapsto \Lambda^\mu{}_\nu x^\nu \tag{3.31}$$

usual transformations are infinitessimal.

We require

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \underbrace{\omega^\mu{}_\nu}_{\text{infinitessimal}} \tag{3.32}$$

Since $\Lambda^T \eta \Lambda = \eta$,

$$(\delta^\mu{}_\lambda + \omega^\mu{}_\lambda)(\delta^\nu{}_\kappa + \omega^\nu{}_\kappa)\eta^{\lambda\kappa} = \eta^{\mu\nu} \qquad (3.33)$$

and you can show that (to order of $\mathcal{O}(\omega)$)

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0. \qquad (3.34)$$

Proof: $(\delta^\mu{}_\lambda + \omega^\mu{}_\lambda)(\delta^\nu{}_\kappa + \omega^\nu{}_\kappa)\eta^{\lambda\kappa}$

$$= \left(\delta^\mu{}_\lambda \delta^\nu{}_\kappa + \delta^\mu{}_\lambda \omega^\nu{}_\kappa + \omega^\mu{}_\lambda \delta^\nu{}_\kappa + \omega^\mu{}_\lambda \omega^\nu{}_\kappa\right)\eta^{\lambda\kappa}$$

$$= \eta^{\mu\nu} + \eta^{\mu\kappa}\omega^\nu{}_\kappa + \omega^\mu{}_\lambda \eta^{\nu\lambda} + \mathcal{O}(\omega^2)$$

$$= \eta^{\mu\nu} + \omega^{\nu\mu} + \omega^{\mu\nu} + \mathcal{O}(\omega^2)$$

But this is all equal to $\eta^{\mu\nu}$, hence

$$0 = \omega^{\nu\mu} + \omega^{\mu\nu} \left(+ \mathcal{O}(\omega^2)\right)$$

as desired. $\blacksquare$

This implies the "matrix" of entries

$$\omega \equiv \begin{bmatrix} 0 & -x & -y & -z \\ x & 0 & -\alpha & -\beta \\ y & \alpha & 0 & -\gamma \\ z & \beta & \gamma & 0 \end{bmatrix} \qquad (3.35)$$

6 infinitesimal conserved currents

Now for fields:

$$\phi_a(x) \longmapsto \phi'_a(x) \equiv \phi_a(\Lambda^{-1} x) \qquad (3.36)$$

Substituting (3.35) into (3.36),

$$\phi(x^\mu \underset{\text{'$-$' for inverse LT}}{-} \omega^\mu{}_\nu x^\nu) \overset{\mathcal{O}(\omega)}{\equiv} \phi_a(x^\mu) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi_a(x^\mu) \qquad (3.37)$$

We can then work out the infinitessimal symmetries

$$\delta\phi_a = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi_a(x^\mu)] \quad X[\phi_a] \qquad (3.38)$$

$$\delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} - \partial_\mu \underbrace{(\omega^\mu{}_\nu x^\nu \mathcal{L})}_{F^\mu} \qquad (3.39)$$

Noether's Theorem:

$$j^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \omega^\mu{}_\nu x^\nu \partial_\lambda \phi_a(x^\lambda) + \omega^\mu{}_\nu x^\nu \mathcal{L} \qquad (3.40)$$

* 6 independent conserved currents (J^μ)

Bringing out the '$\omega^\mu{}_\nu x^\nu$' terms, we find this conserved current to be the difference of two energy momentum tensors, multiplied by x^ν:

$$(J^\mu)^{\rho\nu} = x^\rho T^{\mu\nu} - x^\nu T^{\mu\rho} \qquad (3.41)$$

and

$$\partial_\mu (J^\mu)^{\rho\nu} = 0 \qquad (3.42).$$

What are the conserved charges?

$j,k \in \{1,2,3\}$

$$Q^{jk} = \int d^3x \left(x^j T^{0k} - x^k T^{0j} \right) \qquad (3.43)$$

"Generators of Rotations"

$$Q^{0j} = \int d^3x \left(x^0 T^{0j} - x^j T^{00} \right) \qquad (3.44)$$

"Generators of Boosts"

Why are these charges called generators?

Recall Poisson Bracket:

Suppose (q_j, p_j) $(j = 1, \dots)$ are pairs of canonical coordinates, let $f, g : \underbrace{\mathbb{R}^N \times \mathbb{R}^N}_{\text{phase space}} \longrightarrow \mathbb{R}$. The Poisson Bracket between f, g is

$$\{f, g\} = \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \qquad (3.45).$$

The corresponding Poisson Bracket for fields is

$$\{F, G\} = \int d^3x \frac{\delta F}{\delta \phi}\frac{\delta G}{\delta \pi} - \frac{\delta F}{\delta \pi}\frac{\delta G}{\delta \phi} \qquad (3.46)$$

It turns out $$\frac{df}{dt} = \{f, H\} \qquad (3.47)$$

Replacing H (eq. (3.47)) with a conserved charge (eq. (3.43) / (3.44)),

$$\frac{\partial f}{\partial S^{\lambda\kappa}} = \{f, Q^{\lambda\kappa}\} \qquad (3.48)$$

which generates the corresponding symmetry transformations.

Furthermore, these conserved charges obey lie algebra

$$\{Q^{\rho\sigma}, Q^{\rho'\sigma'}\} = \ldots Q^{\alpha\rho} \qquad (3.49)$$

Lie algebra of Poincaré Group

Standard Dogma of Quantization:

bounded on a Hilbert space

f on phase space $\longrightarrow$ $\hat{f}$ observable

$$\{f, g\} = h \longrightarrow [\hat{f}, \hat{g}] = i\hat{h}$$

$$Q^{\lambda\kappa} \longrightarrow \hat{Q}^{\lambda\kappa}$$

Generators of Lorentz Transformations in the Hilbert Space

$$\frac{d\hat{u}}{ds} = i\left[\hat{u}, \hat{Q}^{\lambda\kappa}\right] \omega_{\lambda\kappa} \qquad (3.50)$$